How Slow, or Fast, are Standard Random Walks?
– Analysis of Hitting and Cover Times on Trees

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Abstract

Random walk is a powerful tool, not only for modeling, but also for practical use such as the Internet crawlers. *Standard random walks* on graphs have been well studied; It is well-known that both *hitting time* and *cover time* of a standard random walk are bounded by $O(n^3)$ for any graph with n vertices, besides the bound is tight for some graphs. Ikeda et al. (2003) provided "β-random walk," which realizes $O(n^2)$ hitting time and $O(n^2 \log n)$ cover times for any graph, thus it archives, in a sense, "n-times improvement" compared to the standard random walk.

This paper is concerned with optimizations of hitting and cover times, by drawing a comparison between the standard random walk and the fastest random walk. We show for any tree that the hitting time of the standard random walk is at most $O(\sqrt{n})$-times longer than one of the fastest random walk. Similarly, the cover time of the standard random walk is at most $O(\sqrt{n \log n})$-times longer than the fastest one, for any tree. We also show that our bound for the hitting time is tight by giving examples, while we only give a lower bound $\Omega(\sqrt{n/\log n})$ for the cover time.

Keywords: Random walk, Markov chain, Hitting time, Cover time.

1 Introduction

1.1 Standard random walk

Given a finite undirected and connected graph $G = (V, E)$ the transition probability matrix P_0 of the *standard random walk* is defined by, for $u, v \in V$,

$$p_{uv} = \begin{cases} \frac{1}{\deg(u)} & v \in N(u), \\ 0 & \text{otherwise,} \end{cases}$$

where $N(u)$ and $\deg(u)$ are the set of vertices adjacent to vertex u and the degree of u respectively.

For a random walk on G with a transition probability P in general, let $H_G(P; u, v)$ denote the hitting time from $u \in V$ to v, that is the expected number of transitions necessary for random walk starting from u to reach v for the first time under P, and let $H_G(P)$ denote the hitting time of G, that is defined by $H_G(P) = \max_{u,v \in V} H_G(P; u, v)$. The cover time $C_G(P; u)$ from u is the expected number of transitions necessary for random walk starting from u to visit all vertices in V under P, and the cover time $C_G(P)$ of G is defined as $C_G(P) = \max_{u \in V} C_G(P; u)$.

Let $n = |V|$ and $m = |E|$. Then the cover time of the standard random walk of any graph G with n vertices and m edges holds $C_G(P_0) \leq 2m(n-1)$ (leliunas et al., 1979; Aldous, 1983), whose results were later refined by Feige (1995) as

$$(1 - p(1))n \log n \leq C_G(P_0) \leq (1 + o(1))\frac{4}{27}n^3.$$

There is a graph L (called a Lollipop) such that

$$H_L(P_0) = (1 - o(1))\frac{4}{27}n^3,$$

and $C_L(P_0) \geq H_L(P_0)$, and thus both the hitting and the cover times of standard random walks are $\Theta(n^3)$ (Gilks et al., 1995).

1.2 Related work

Ikeda et al. (2003) proposed a random walk with a transition probability matrix $P_1 = (p_{uv})_{u,v \in V}$ defined by

$$p_{uv} = \begin{cases} \frac{\deg(u)^{-1/2}}{\sum_{w \in N(u)} \deg(w)^{-1/2}} & u \in N(u), \\ 0 & \text{otherwise.} \end{cases}$$

for any $u, v \in V$, and showed for any graph G with order n that the hitting and the cover times of the random walk are bounded by $O(n^2)$ and $O(n^2 \log n)$, respectively. In addition, they proved that the hitting and the cover times of any random walk on path graph are bounded by $\Omega(n^2)$. It should be noted that the above random walk is defined only by using local degree information.

When we are allowed to use (global) information on G to define a transition probability, instead of local degree information, we may obtain a faster random walk. In fact, it is easy to see that, for any graph G, there exists a random walk whose cover time is $O(n^2)$, by considering a random walk on its spanning tree.

It remains to be seen if there exists a random walk using "local information" only, such that its cover time is $O(n^2)$ for any graph. (Ikeda et al., 2003)

1.3 Results

Our motivation is an optimization of the hitting and the cover times of random walks on graphs by designing a transition probability matrix. Generally, there is a graph which has n-times faster random walk than the standard random walk. For example, The hitting and the cover times of β-random walk on lollipop graph where $\beta = 1/2$ are both $\Theta(n^2)$ while ones of the

standard random walk are $\Theta(n^3)$. On the other hand, the standard random walk is the fastest one on some graphs(e.g., path graph). In this paper, we consider that the optimization of random walks on trees.

In Section 2, we give the definition of random walk on graph and present some early studies. In Section 3, we show that for any random walk on any tree, the cover is bounded by $\Omega(n \log n)$. Then, we investigate the ratio between the standard random walk and the optimal random walk of the hitting and the cover times. Let P^* be a transition probability matrix of which the hitting or the cover time becomes minimum. For any tree T, the ratio of $\frac{H_G(P_0)}{H_G(P^*)}$ and $\frac{C_G(P_0)}{C_G(P^*)}$ has smaller upper bound than n. In Section 4, we show $\frac{H_T(P_0)}{H_T(P^*)} = \mathrm{O}(\sqrt{n})$, which is tight. In Section 5, we show $\frac{C_T(P_0)}{C_T(P^*)} = \mathrm{O}(\sqrt{n \log n})$ using the results in Section 3 and 4.

2 Preliminaries

Suppose a graph $G = (V, E)$ is finite, undirected, simple and connected with the order $n = |V|$ and the size $m = |E|$. Let l be a diameter of G. For $u \in V$, let $N(u) = \{v \mid (u, v) \in E\}$ and $\deg(u) = |N(u)|$ be the set of vertices adjacent to a vertex $u \in V$ and the degree of u, respectively.

We say $P = (p_{uv})_{u,v \in V}$ is a transition probability matrix of a random walk on G if $p_{uv} > 0$ for $(u, v) \in E$, $p_{uv} = 0$ for $(u, v) \notin E$ and $\sum_{v \in V} p_{uv} = 1$.

Assume that each undirected edge $e = (u, v)$ has a weight w_{uv}. We can obtain a *weighted walk* whose transition probability is defined by

$$p_{uv} = \frac{w_{uv}}{\sum_{v' \in N(u)} w_{uv'}}.$$

If all weights are 1, the weighted walk is the standard random walk.

We assume that each edge $(u, v) \in E$ is a resistance whose value is $r_{uv} = 1/w_{uv}$. For any u, v, let R_{uv} be the effective resistance between u and v. Let $w = \sum_{(u,v) \in E} w_{uv}$.

Theorem 1 *(Chandora et al., 1997) Arbitrarily given weighted graph G, if a transition probability matrix P is defined by the edge weight, then $H_G(P; u, v) + H_G(P; u, v) = 2wR_{uv}$ for any $u, v \in V$.*

Any irreducible random walk on tree is represented as a weighted walk. From Theorem 1, we obtain the following.

Proposition 2 *Assume that the shortest path from u to v is $u = v_1, v_2, \cdots, v_k = v$. For any u, v on a tree T,*

$$H_T(P; u, v) + H_T(P; v, u) = 2w \sum_{i=1}^{k-1} \frac{1}{w_{v_i, v_{i+1}}}.$$

Proof. Since T is a tree, R_{uv} depends only on the sum of the edge resistance on a unique path from u to v. Therefore $R_{uv} = \sum_{i=1}^{k-1} \frac{1}{w_{v_i, v_{i+1}}}$. $\qquad \square$

Theorem 3 $l(n-1) \leq H_T(P_0) \leq 2l(n-1)$.

Proof. In the standard random walk, all of the edge weights are 1. Thus $H_T(P_0; u, v) + H_T(P_0; v, u) \leq 2l(n-1)$. For any $u, v \in V$, and the equality holds when u and v defines the diameter l. Thus $\max\{H_T(P_0; u, v), H_T(P_0; v, u)\} \leq l(n-1)$ for u, v with the distance l. $H_T(P_0) \leq 2l(n-1)$ is clear. $\square$

The following Matthew's theorem (Ikeda et al., 2003; Matthews, 1988) is known, which bounds the cover time by the hitting time.

Theorem 4 For any G and P,

$$C_G(P) \geq \min_{u,v \in V} H_G(P; u, v) \cdot h(n-1),$$
$$C_G(P) \leq \max_{u,v \in V} H_G(P; u, v) \cdot h(n-1),$$

holds, where $h(n)$ is the n-th harmonic number, which is defined by $h(n) = \sum_{i=1}^{n} \frac{1}{i}$, that is $h(n)$ is almost $\ln n$.

3 Lower bounds of cover times on trees

In this section, we prove that for any tree with n vertices and for any transition probability matrix, the cover time of a random walk is $\Omega(n \log n)$, which we will use in Section 5 to derive an upper bound of $\frac{C_G(P_0)}{C_G(P^*)}$. First, we show that the tight lower bound of the cover time of a random walk on the star graph is $n \cdot h(n) \approx n \log n$. Then, we show that the cover time of the fastest random walk on an arbitrarily fixed tree with n vertices is not less than the cover time of the fastest random walk on the star graph with n vertices.

3.1 Lower bound on the cover time of a star

A star graph is a tree whose diameter is 2. Let a star graph with n vertices be denoted by $S_n = (\{u_1, \cdots, u_n\}, \{(u_1, u_i) \mid i = 2, \cdots, n\})$, in which $n-1$ vertices are connected to a vertex u_1 as pendant-vertices. We first show that for any random walk on S_n, the cover time is $\Omega(n \log n)$.
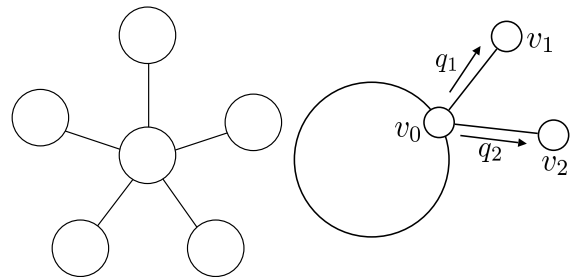

Figure 1: Star graph

Figure 2: A graph with two pendant-vertices

To prove this, we consider transition probabilities between vertex v_0 and two pendant-vertices v_1, v_2 connected to v_0 in a general setting. Let $G = (V, E)$ be a graph that has two pendant-vertices v_1 and v_2. These pendant-vertices are both connected to vertex v_0. Let P be a transition probability matrix of a random walk on G, and p_1 and p_2 denote the transition probabilities from v_0 to v_1 and from v_0 to v_2, respectively. Fix the probabilities in P other than p_1 and p_2, and we consider another transition probability $P(q_1, q_2)$ in which p_1 and p_2 are replaced by q_1 and q_2 satisfying that $0 \leq q_1 \leq 1$, $0 \leq q_2 \leq 1$ and $q_1 + q_2 = p_1 + p_2$.

Lemma 5 *The cover time $C_G(P(q_1, q_2))$ is minimized by q_1, q_2 satisfying that $q_1 = q_2$.*

Proof. Let $V' = V \setminus \{v_1, v_2\}$, and let p_i be the transition probability from v_0 to $w_i \in \{N(v_0) \cap V'\}$ for $i = 3, \cdots \deg(v_0)$.

The hitting times from v_0 to pendant-vertices v_1 and v_2 are represented as follows:

$$H_G(P(q_1, q_2); v_0, v_1) = \frac{1}{q_1}(1 + q_2 + \tilde{H}), \quad (1)$$

$$H_G(P(q_1, q_2); v_0, v_2) = \frac{1}{q_2}(1 + q_1 + \tilde{H}), \quad (2)$$

where $\tilde{H} \stackrel{\text{def}}{=} \sum_{i=3}^{\deg(v_0)} p_i H_G(P(q_1, q_2); w_i, v_0)$ and p_i denotes the transition probability of P from v_0 to w_i. We consider the following two cases:

(i) **Starting at** $u \in V'$

Hitting times between two vertices other than v_1 and v_2 do not depend on q_1 and q_2. This implies that the expected number of transitions of the token to visit all vertices other than v_1 and v_2 is fixed. Thus, the cover time is minimized when the expected number of transitions for the token to visit both v_1 and v_2 achieves the minimum. Let C' denote the expected number of transitions when the token starting from v_0 takes to visit v_1 and v_2. We consider three cases on covering process of v_1 and v_2. First, if the token visits v_1 with probability q_1 in the first step, then it visits v_2 with $1 + H(v_0, v_2)$ expected steps. Second, if the token visits v_2 with probability q_2 in the first step, then it visits v_1 with $1 + H(v_0, v_2)$ expected steps. Finally, for $i = 3, \ldots, \deg(v_0)$, if the token visits w_i with probability p_i in the first step and return v_0 with $H(w_i, v_0)$ expected steps, then it visits v_1 and v_2 with C' expected steps. Therefore C' is represented by

$$C' = q_1(2 + H(v_0, v_2)) + q_2(2 + H(v_0, v_1))$$
$$+ \sum_{i=3}^{\deg(v_0)} p_i(1 + H(w_i, v_0) + C')$$
$$= 2 + \frac{(q_1^2 + q_2^2)(1 + \tilde{H}) + q_1^3 + q_2^3}{(1 - \tilde{p})q_1 q_2} + \frac{\tilde{p} + \tilde{H}}{1 - \tilde{p}}, \quad (3)$$

where $\tilde{p} \stackrel{\text{def}}{=} \sum_{i=3}^{\deg(v_0)} p_i$. In the following, we consider to find q_1 and q_2 minimizing C', meaning that the cover time $C(P(q_1, q_2); u)$ is minimum. Since $q_1 + q_2 = 1 - \tilde{p}$ by the definition of $\tilde{p}$, we have

$$q_1^2 + q_2^2 = (q_1 + q_2)^2 - 2q_1 q_2 = (1 - \tilde{p})^2 - 2q_1 q_2,$$
$$q_1^3 + q_2^3 = (1 - \tilde{p})((1 - \tilde{p})^2 - 3q_1 q_2).$$

Since $\tilde{p}$ is fixed, Equation (3) is minimized when $q_1 q_2$ takes the maximum, i.e., $q_1 = q_2$.

(ii) **Starting at** $u \in \{v_1, v_2\}$

We abbreviate $H(P(q_1, q_2); v_0, v_1)$ and $H(P(q_1, q_2); v_0, v_2)$ to $H(v_0, v_1)$ and $H(v_0, v_2)$, respectively. The cover time $C(P(q_1, q_2; u))$ is minimized when $\max\{H(v_0, v_1), H(v_0, v_2)\}$ takes the minimum. Let q_m be $\max\{q_1, q_2\}$, then

$$\max\{H(v_0, v_1), H(v_0, v_2)\} = \frac{1 + q_m + \tilde{H}}{1 - \tilde{p} - q_m} \quad (4)$$

holds. Equation (4) is minimized when q_m takes the minimum, that is $q_1 = q_2$. $\square$

Lemma 5 can be generalized to the case in which v_0 is connected with k pendant-vertices for any positive integer $k \geq 2$. Let $G = (V, E)$ be a graph that has k pendant-vertices $v_1, \cdots, v_k$, and all of these pendant-vertices are connected to vertex v_0. Given a transition probability matrix P on G, we can define $P(q_1, q_2, \ldots, q_k)$ similarly to $P(q_1, q_2)$, that is, $P(q_1, q_2, \ldots, q_k)$ is the transition probability matrix obtained from P by replacing the transition probabilities from v_0 to v_i in P with a variable q_i. Then we have the following.

Lemma 6 *The cover time $C_G(P(q_1, q_2, \ldots, k))$ is minimized at $q_1 = \cdots = q_k$.*

By applying Lemma 6 to a star graph, we can see that the standard random walk is the fastest random walk for star graph in terms of the cover time. Since the cover time of the standard random walk on the star graph can be evaluated as the coupon collector problem, we obtain the following:

Theorem 7 *The cover time of the standard random walk on star graph S_n is $2n \cdot h(n-1)$, where $h(i)$ is the i-th harmonic number.*

Corollary 8 *For any random walk of transition probability matrix P on star graph S_n with n vertices, $C_{S_n}(P) = \Omega(n \log n)$.*

3.2 Comparison of cover times on trees

As mentioned at the beginning of Section 3, we will show in this subsection that S_n is the graph that has the fastest cover time random walk among the trees of n vertices. To this end, we first present a way to "compare" two graphs having a similar structure.

Let us consider two graphs $G_A = (V, E_A)$ and $G_B = (V, E_B)$, which have almost the same structure described as follows. Suppose that G_A and G_B have a common maximal induced sub graph, say $G_0 = (V_0, E_0)$. Both the induced sub graphs of G_A and G_B on $V \setminus V_0 \cup \{x\}$ are isomorphic and form the star graph $S_{k+1} = (\{v_1, \ldots, v_{k+1}\}, \{(v_1, v_i) \mid i = 1, \ldots, k+1\})$. Only the difference is that $x = v_{k+1}$ in G_A but $x = v_1$ in G_B. Here we rename the stars of G_A and G_B as follows:

$$S_A = (\{y, z_1, \cdots, z_k\}, \{(y, x)\} \cup \{(y, z_i) \mid i = 1, \ldots, k\}),$$
$$S_B = (\{y, z_1, \ldots, z_k\}, \{(y, x)\} \cup \{(x, z_i) \mid i = 1, \ldots, k\}).$$

Lemma 9 *For any transition probability matrix P for G_A and $u \in V_0$, there is a transition probability matrix P' for G_B such that $C_{G_A}(P; u) \geq C_{G_B}(P'; u)$.*
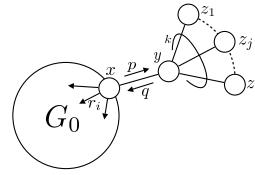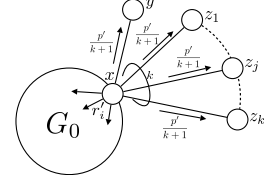

Figure 3: G_A Figure 4: G_B

Proof. In order to show the lemma, it is sufficient to consider the case when $P = (p_{uv}^{(A)})_{u,v \in V}$ on G_A gives the minimum cover time. Thus we assume that transition probabilities in P to pendant-vertices are uniform by Lemma 6. Let $p_{xy}^{(A)} = p, p_{yx}^{(A)} = q$, for simplicity.

$$p_{yz_1}^{(A)} = p_{yz_2}^{(A)} = \cdots = p_{yz_k}^{(A)} = \frac{1 - q}{k}.$$

For this P, we construct $P' = (p_{uv}^{(B)})_{u,v \in V}$ as follows:

$$p_{uv}^{(B)} = \begin{cases} p_{uv}^{(A)} & u, v \in V_0 \setminus \{x\}, \\ \gamma p_{xv}^{(A)} & u = x, \text{ and } v \in V_0, \\ p'/(k+1) & u = x, \text{ and } v \in \{y, z_1, \ldots, z_k\}, \\ 1 & v = x, \text{ and } u \in \{y, z_1, \ldots, z_k\}, \end{cases}$$

where p' and $0 \leq \gamma \leq 1$ are tuned later.

From now on, we show that there exists P' satisfying that

$$C_{G_A}(P; u) \geq C_{G_B}(P'; u)$$

with tuned P' and γ.

For G_A and P, let ρ be the expected number of steps that it takes for the token starting from x to visit a vertex in $V_0 \setminus \{x\}$. Then we have

$$\rho = \frac{2p}{q(1-p)}. \qquad (5)$$

Similarly, we define ρ' for G_B and P', then P' satisfies that

$$\rho' = \frac{2p'}{1-p'}. \qquad (6)$$

Since G_0 and its transition probability are common for (G_A, P) and (G_B, P'), it suffices to show $\rho \geq \rho'$ for the claim. Let s be the expected number of times that the token visits x to cover the vertices in $V \setminus V_0$ in G_A and P. We can see that

$$s\rho = \frac{2k \cdot h(k)}{1-q} + \frac{2}{q},$$
$$= 2\left(k \cdot h(k) + 1\right) + 2\left(\frac{q \cdot k \cdot h(k)}{1-q} + \frac{1-q}{q}\right).$$

For this s, we can set p' and γ such that

$$s\rho' = 2(k+1) \cdot h(k+1),$$
$$= 2(k+1)\left(h(k) + \frac{1}{k+1}\right),$$
$$= 2(k \cdot h(k) + 1) + 2h(k),$$

Then this implies that $\rho \geq \rho'$; If $\frac{q \cdot k}{1-q} \geq 1$, clearly $\rho > \rho'$. On the other hand, if $\frac{q \cdot k}{1-q} < 1$, also $\rho > \rho'$, since $k < \frac{1-q}{q}$ and $h(k) < k$. Thus we have $C_{G_A}(P) \geq C_{G_B}(P')$. $\qquad \square$

By applying Lemma 9 to a tree T repeatedly, we obtain the following theorem and corollary.

Theorem 10 *For any tree T with n vertices and any transition probability matrix P on T, $C_T(P) \geq C_{S_n}(P_0)$ holds, where P_0 is the transition probability matrix of the standard random walk on S_n.*

Corollary 11 *The cover time of any random walk for any tree is $\Omega(n \log n)$.*

4 Limitation of speeding up on hitting time

In this section, we show for any tree T that the ratio of the hitting time of standard random walk $H_T(P_0)$ and optimal hitting time $H_T(P^*)$ is upper bounded by $O(\sqrt{n})$.

To begin with, we show the following:

Lemma 12 *For any tree T and transition matrix P, $H_T(P) \geq \max\{n-1, l^2\}$.*

Proof. Without loss of generality we may assume $w = n-1$. Then, there exists an edge whose weight is at most one since the number of edges is $n-1$. Therefore, maximum resistance $R_{\max}$ is at least one. Hence, we obtain $H_T(P) \geq n-1$ from Proposition 2.

Since the diameter is l, it is clear that $H_T(P) \geq l^2$.

These inequalities are satisfied coincidentally. The hitting time is at least $\max\{n-1, l^2\}$. $\qquad \square$

Theorem 13 *For any tree T, $\frac{H_T(P_0)}{H_T(P^*)} \leq 2\sqrt{n-1}$.*

Proof. (i)If $l \geq \sqrt{n-1}$, then

$$\frac{H_T(P_0)}{H_T(P^*)} \leq \frac{2l(n-1)}{l^2} \leq 2\sqrt{n-1}$$

holds, since $H_T(P^*) \geq l^2$.
(ii)If $l < \sqrt{n-1}$, then

$$\frac{H_T(P_0)}{H_T(P^*)} \leq \frac{2l(n-1)}{n-1} < 2\sqrt{n-1}$$

holds, since $H_T(P^*) \geq n-1$.

In both cases, $\frac{H_T(P_0)}{H_T(P^*)} \leq 2\sqrt{n-1}$ holds. $\qquad \square$

In the rest of this section, we show a tight example of speeding up of the hitting time. That is a broom graph whose diameter is $\Theta(\sqrt{n})$.

Definition 14 *A broom graph $B_{k,l} = (V, E)$ is defined by $V = V_T \cup V_S$ and $E = E_T \cup E_S$, where*

$$V_T = \{v_i \mid 1 \leq i \leq l+1\},$$
$$V_S = \{u_j \mid 1 \leq j \leq k\},$$
$$E_T = \{(v_i, v_{i+1}) \mid 1 \leq i < l\},$$
$$E_S = \{(v_{l+1}, u_j) \mid 1 \leq j \leq k\}.$$

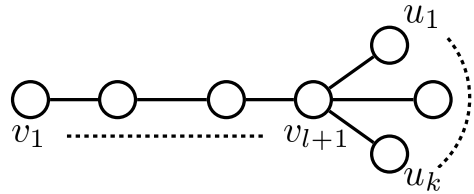Figure 5 is example of broom graph $B_{3,3}$.

Figure 5: Broom graph $B_{3,3}$

Proposition 15 *For a broom graph $B_{k,l}$ with $l = \sqrt{n-1}, k = n-l-1$, there exist a random walk P_1 such that $H_{B_{k,l}}(P_0)/H_{B_{k,l}}(P_1) \geq \sqrt{n-1}/8$.*

Proof. We define edge weights of $B_{k,l}$ by

$$w_e = \begin{cases} 1 + \frac{k}{2l} & e \in E_T, \\ \frac{1}{2} & e \in E_S, \end{cases}$$

then the total weight w is $n-1$. At that time, $R_{\max}$ is $R_{u_1 v_{l+1}} + \max\{R_{v_1 v_{l+1}}, R_{u_2 v_{l+1}}\}$. Now,

$$R_{u_1 u_{l+1}} = \frac{l}{1 + \frac{k}{2l}}$$

and $R_{u_1 v_{l+1}} = R_{u_2 v_{l+1}} = \cdots = R_{u_k v_{l+1}} = 2$, respectively.

Now set $l = \sqrt{n-1}$, then $R_{\max} = 4$ since

$$R_{v_0 u_{l+1}} = \frac{2(n-1)}{n-1+\sqrt{n-1}} < 2 = R_{u_2 v_{l+1}}.$$

From Proposition 2, $H(P') \leq 2wR_{\max} \leq 8(n-1)$. From Theorem 3, $H(P_0) \geq (n-1)^{1.5}$. Thus, we obtain that

$$\frac{H(P_0)}{H(P_1)} \geq \frac{(n-1)^{1.5}}{8(n-1)} = \frac{\sqrt{n-1}}{8}.$$

$\qquad \square$

5 Limitation of speeding up on cover time

In this section, we provide that for any tree T, the ratio of the cover time of standard random walk $C_T(P_0)$ and optimal cover time $C_T(P^*)$ is upper bounded by $O(\sqrt{n})$.

Lemma 16

(1) $C_T(P_0) \leq \min\{2(n-1)^2, 2l(n-1)\log(n-1)\}.$

(2) $C_T(P^*) \geq \max\{l^2, 2(n-1)\log(n-1))\}.$

Proof. (1). For any graph, the cover time of a standard random walk is upper bounded by $2m(n-1)$ (leliunas et al., 1979; Aldous, 1983). Since a tree has just $n-1$ edges, the cover time is at most $2(n-1)^2$. On the other hand, Theorems 3 and 4 indicate that $2l(n-1)\log(n-1)$ is an upper bound of the cover time of the standard random walk.

(2) Since the hitting time can be a lower bound of the cover time clearly, Lemma 12 gives a lower bound $\max\{n-1, l^2\}$ of the cover times. On the other hand, Theorem 10 gives another lower bound $C_S(P_0) = 2(n-1)\log(n-1)$ of the cover time of any random walk on any tree. $\square$

Theorem 17 For any tree T,

$$\frac{C_T(P_0)}{C_T(P^*)} \leq \sqrt{2(n-1)\log(n-1)}.$$

Proof. From Lemma 16,

$$\frac{C_T(P_0)}{C_T(P)} \leq \frac{\min\{2(n-1)^2, 2l(n-1)\log(n-1)\}}{\max\{2(n-1)\log n, l^2\}}$$

holds.

(i) If $2l\log(n-1) \leq n-1$ and $2(n-1)\log(n-1) \leq l^2$, then

$$\frac{C_T(P_0)}{C_T(P)} \leq \frac{2l(n-1)\log(n-1)}{l^2},$$
$$\leq \sqrt{2(n-1)\log(n-1)},$$

holds.

(ii) If $2l\log(n-1) \leq n-1$ and $2(n-1)\log(n-1) > l^2$, then

$$\frac{C_T(P_0)}{C_T(P)} \leq \frac{2l(n-1)\log(n-1)}{2(n-1)\log(n-1)},$$
$$< \sqrt{2(n-1)\log(n-1)},$$

holds.

(iii) If $2l\log(n-1) > n-1$ and $2(n-1)\log(n-1) \leq l^2$, then

$$\frac{C_T(P_0)}{C_T(P)} \leq \frac{2(n-1)^2}{l^2},$$
$$< 4\log^2(n-1), \qquad (7)$$

holds. In case that n is large enough, Equation (7) is smaller than $\sqrt{2(n-1)\log(n-1)}$.

(iv) If $2l\log(n-1) > n-1$ and $2(n-1)\log(n-1) > l^2$, then

$$n-1 < 2l\log(n-1),$$
$$< \sqrt{n-1}(2\log(n-1))^{1.5},$$

holds. This inequality implies that $n-1 \leq \log^3(n-1)$. Hence case (iv) does not appear when n is large enough.

Therefore, $\frac{C_T(P_0)}{C_T(P^*)} \leq \sqrt{2(n-1)\log(n-1)}$ in all cases. $\square$

6 Conclusion

In this paper, we investigated speeding up random walk on trees. We first proved that for any tree with n vertices and for any transition probability matrix, the cover time is $\Omega(n\log n)$. Then, we presented the hitting and the cover time ratio of standard random walk to optimal random walk is $O(\sqrt{n})$ and $O(\sqrt{n\log n})$, respectively. In particular, the ratio of hitting time is tight. Whether that the standard random walk is slow or fast, bad or good depends on the intended use.

There still remains some problems. The tight case of the cover time is not found. We would like to design better random walks than the standard random walk whose ratio compered to optimum are $o(\sqrt{n})$ for the hitting time and $o(\sqrt{n\log n})$ for the cover time. An extension of arguments to more generally graph is a future work. It is open if there always exists a fastest walk for hitting time which is also fastest for cover time, and vice versa.

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