Pancycles and Hamiltonian-Connectedness of the Hierarchical Cubic Network

Jung-Sheng Fu
Takming College, Taipei, TAIWAN
jsfu@mail.takming.edu.tw
Gen-Huey Chen
Department of Computer Science and Information Engineering,
National Taiwan University, Taipei, TAIWAN
ghchen@csie.ntu.edu.tw

Abstract

We show that the hierarchical cubic network, an alternative to the hypercube, is hamiltonian-connected using Gray codes. A network is hamiltonian-connected if it contains a hamiltonian path between every two distinct nodes. In other words, a hamiltonian-connected network can embed a longest linear array between every two distinct nodes with dilation, congestion, load, and expansion equal to one. We also show that the hierarchical cubic network contains cycles of all possible lengths but three and five. Since the hypercube contains cycles only of even lengths, it is concluded that the hierarchical cubic network is superior to the hypercube in hamiltonicity. Our results can be applied to the hierarchical folded-hypercube network as well.

Keywords: Embedding, Gray code, hamiltonian-connected, hierarchical cubic network, hierarchical folded-hypercube network, hypercube, pancyclic.

1 Introduction

The hierarchical cubic network (HCN for short), which was proposed by Ghose and Desai (1995) as an alternative to the hypercube, consists of 2^n basic components, named *clusters*. Each cluster is an n-dimensional hypercube (n-cube for short). If each cluster is viewed as a single node, then the HCN appears as a 2^n-node complete graph. The HCN can emulate a comparable hypercube (i.e. a hypercube of the same number of nodes) in constant time, but with only about half as many links per node. The average internode distances in the HCN under random and localized traffic patterns are the same as a comparable hypercube. When message generation rates are moderate, the average message transit delays in the HCN are slightly better than a comparable hypercube. This is a consequence of the fact that the HCN has a smaller maximum routing distance than a comparable hypercube.

Chang and Chen (1996) and Yun and Park (1995, 1998) have derived a shortest-path routing algorithm for the HCN. Chang and Chen (1996) have demonstrated a broadcasting algorithm. Ghose and Desai (1995) have developed some parallel algorithms. Yun and Park (1995, 1998) have shown that the diameter is about two-thirds the diameter of a comparable hypercube. Chiang and Chen (1996) and Yun and Park (1995) have constructed hamiltonian cycles. Fu and Chen (2001) have derived that the wide-diameter and fault-diameter are also about two-thirds those of a comparable hypercube.

Suppose that W is an interconnection network (network for short). The *pancycle problem* on W asks, for every integer $3 \le l \le |W|$, whether or not W contains a cycle of length l, where $|W|$ is the number of nodes contained in W. The pancycle problem was solved on a lot of networks, e.g., the twisted cube (Chang, Wang, and Hsu 1999), the butterfly graph (Hwang and chen 2000), the hyper Petersen network (Das, Öhring, and Bganerjee 1995), the folded Petersen cube network (Öhring and Das 1996), the arrangement graph (Day and Tripathi 1993), the hypercomplete network (Chen, Fu, and Fang 2000), and the alternating group graph (Jwo, Lakshmivarahan, and Dhall 1993). If W contains cycles of lengths ranging from three to $|W|$, then W is called *pancyclic*. The arrangement graph (Day and Tripathi 1993), the hypercomplete network (Chen, Fu, and Fang 2000), the supercube (Auletta, Rescigno, and Scarano 1995), and the alternating group graph (Jwo, Lakshmivarahan, and Dhall 1993) were shown pancyclic. Throughout this paper, network and graph are used interchangeably.

A path in W is called a *hamiltonian path* if it contains every node of W exactly once. W is called *hamiltonian-connected* if there is a hamiltonian path between every two distinct nodes of W. The hypercomplete network (Chen, Fu, and Fang 2000), the alternating group graph (Jwo, Lakshmivarahan, and Dhall 1993), and the arrangement graph (Lo and Chen 2001) were shown hamiltonian-connected. Apparently, a bipartite graph is not hamiltonian-connected. Instead, Wong (1995) introduced the concept of hamiltonian-laceability for bipartite graphs.

A bipartite graph $G=(V_1, V_2, E)$ with $|V_1|=|V_2|$ is called *hamiltonian-laceable* if there is a hamiltonian path between every node of V_1 and every node of V_2, where V_1 and V_2 are the two partite sets of G. Further, G is *strongly hamiltonian-laceable* if it has the additional property that there is a path of length $|V_1|+|V_2|-2$ between every two distinct nodes of the same partite set (Hsieh, Chen, and Ho 2000). Wong (1995) and Hsieh, Chen, and Ho (2000) have shown that the butterfly graph and the star graph are hamiltonian-laceable and strongly hamiltonian-laceable, respectively. Following conventional usage, every path (or cycle) in this paper contains no repeated node.

We first solve the pancycle problem on the HCN. The HCN contains cycles of all possible lengths but three and five. Then we show that the HCN is hamiltonian-connected. According to our results, the HCN can embed rings of all possible lengths but three and five, and a longest linear array between every two distinct nodes, all with dilation, congestion, load, and expansion equal to one. Since the hypercube is bipartite, every cycle it contains has even length. Hence, it can be concluded that the HCN is superior to the hypercube in hamiltonicity. Our results can be applied to the hierarchical folded-hypercube network (Duh, Chen, and Fang 1995) as well.

In the next section, the HCN is formally defined in graph-theoretic terms. Besides, Gray codes are reviewed and some fundamental properties are derived. They will be used in subsequent cycle construction and path construction.

2 Preliminaries

We denote the HCN containing 2^n clusters, where each cluster is an n-cube, HCN(n). HCN(n) can be defined:

Definition 1. The node set of the HCN(n) is $\{(X, Y) \mid X$ and Y are two binary sequences of length $n\}$. Each node (X, Y) is adjacent to:

(1) $(X, Y^{(k)})$ for all $1 \le k \le n$, where $Y^{(k)}$ is an adjacent node of Y in an n-cube which differs from Y at the kth bit position (from the left),

(2) (Y, X) if $X \ne Y$, and

(3) $(\overline{X}, \overline{Y})$ if $X=Y$, where $\overline{X}$ and $\overline{Y}$ are the bitwise complements of X and Y, respectively.

The cluster where a node (X, Y) resides is denoted by X, and its location in the cluster is denoted by Y. Links (1) are contained within clusters, whereas links (2) and (3) connect two clusters. Links (2) and (3) are referred to as *nondiameter links* and *diameter links*, respectively. The HCN(n) is regular of degree n+1. Figure 1 shows the HCN(3). Unless specified otherwise, we assume $n \ge 3$ throughout this paper.

Conveniently, an n-cube can be represented with $\overbrace{**...*}^{n} = *^n$, where $* \in \{0, 1\}$. Hence, $*^{n-1}1$ and $*^{n-1}0$, which contain the nodes with rightmost bits 1 and 0, respectively, represent two disjoint (n–1)-cubes. We use $d_H(X, Y)$ to denote the *Hamming distance* between X and Y, which is the number of different bits between X and Y. A path from X to Y is abbreviated to an X-Y *path*. Malluhi and Bayoumi (1994) showed that there is a hamiltonian X-Y path in an n-cube if $d_H(X, Y)$ is odd. The result can be generalized as follows.

Lemma 1. Suppose $d_H(X, Y)=d \ge 1$. There are X-Y paths in an n-cube whose lengths are d, d+2, d+4, …, c, where $n \ge 1$, $c=2^n-1$ if d is odd, and $c=2^n-2$ if d is even.

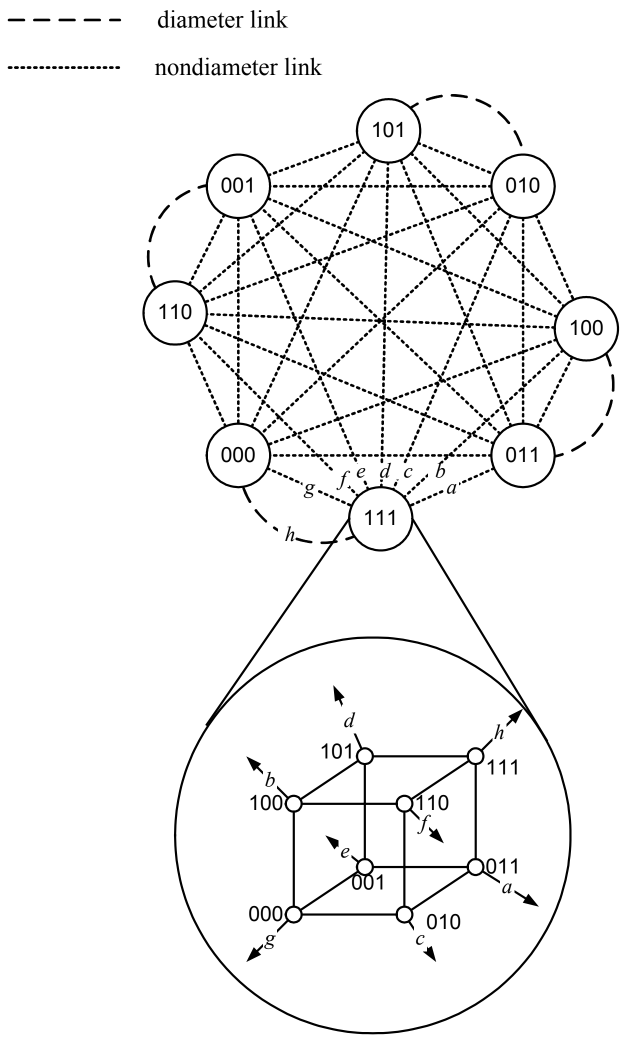

Figure 1. HCN(3)

Proof. We proceed by induction on n. Clearly, the lemma holds for n=1, 2. Assume that it holds for some $k \ge 2$. For $n=k+1$:

Without loss of generality, suppose $X=x_1x_2…x_k0$ and $Y=y_1y_2…y_k1$. We let $Z=z_1z_2…z_k1$ with $d_H(Y, Z)=1$. Then, $d_H(X, Z^{(k+1)})=d_H(X^{(k+1)}, Z)=d_H(X^{(k+1)}, Y)\pm1=d_H(X, Y)-1\pm1 \in \{d-2, d\}$. By our assumption, there are Z-Y paths of lengths 1, 3, …, 2^k-1 in $*^k1$, and X-$Z^{(k+1)}$ paths of lengths d, d+2, …, c' in $*^k0$, where $c'=2^k-1$ if d is odd and $c'=2^k-2$ if d is even.

Now let us consider X-Y paths as follows: $X \Rightarrow Z^{(k+1)} \rightarrow Z \Rightarrow Y$. Throughout this paper, we use $\rightarrow$ to denote a link and $\Rightarrow$ to denote a path (inside a hypercube). The X-Y paths have lengths d+2, d+4, …, c, where $c=2^{k+1}-1$ if d is odd and $c=2^{k+1}-2$ if d is even. The shortest X-Y path has length d. This completes the proof. ☐

As a consequence of Lemma 1, the hypercube is strongly hamiltonian-laceable.

Lemma 2. Suppose that A, B, X, Y are four distinct nodes of an n-cube and $d_H(A, B)=d_H(X, Y)=1$, where $n \ge 2$. There are disjoint A-B and X-Y paths with all lengths 1, 3, 5, …, $2^{n-1}-1$.

Proof. Suppose $A=a_1a_2\ldots a_n$, $B=b_1b_2\ldots b_n$, $X=x_1x_2\ldots x_n$, and $Y=y_1y_2\ldots y_n$. There exists $1\le k\le n$ so that $a_k=b_k=\overline{x_k}=\overline{y_k}$. Hence, $A\in *^{k-1}a_k*^{n-k}$, $B\in *^{k-1}a_k*^{n-k}$, $X\in *^{k-1}\overline{a_k}*^{n-k}$, and $Y\in *^{k-1}\overline{a_k}*^{n-k}$. This lemma holds because of Lemma 1.□

A *Gray code* (Chen and Shin 1987) of n bits contains 2^n distinct code words, denoted by $G(0)$, $G(1)$, …, $G(2^n-1)$, so that each code word is an n-bit binary sequence and $G(i)$ differs from $G((i+1) \bmod 2^n)$ at exactly one bit position, where $0\le i\le 2^n-1$. For example, 000, 001, 011, 010, 110, 111, 101, and 100 constitute a Gray code of three bits. There is a recursive method to generate a Gray code as follows. Initially, let $G(0)=0$ and $G(1)=1$ be a Gray code of one bit. For $n\ge 1$, a Gray code of $n+1$ bits can be generated as $0G(0)$, $0G(1)$, …, $0G(2^n-1)$, $1G(2^n-1)$, …, $1G(1)$, $1G(0)$. A Gray code thus obtained is called a *reflected Gray code*. In the rest of this paper, we use $G^R(0)$, $G^R(1)$, …, $G^R(2^n-1)$ to denote a reflected Gray code of n bits. It is not difficult to see $G^R(0)=0^n$, $G^R(2^{n-1}-2)=010^{n-3}1$, $G^R(2^{n-1}-1)=010^{n-2}$, $G^R(2^n-2)=10^{n-2}1$, and $G^R(2^n-1)=10^{n-1}$. Moreover, they possess the following properties.

(P1) The leftmost bit of $G^R(i)$ is 0 if $0\le i\le 2^{n-1}-1$, and 1 if $2^{n-1}\le i\le 2^n-1$.

(P2) $d_H(G^R(2^{n-1}), G^R(2^n-1))=1$.

(P3) $d_H(G^R(i), G^R(2^j-1-i))=1$ for all $0\le i\le 2^j-1$ and $1\le j\le n$.

(P4) $d_H(G^R(0), G^R(2^k-2))=2$ for all $2\le k\le n$.

We define $S_{u,v}(x_1x_2\ldots x_n)=x_1\ldots x_{u-1}x_vx_{u+1}\ldots x_{v-1}x_ux_{v+1}\ldots x_n$, where $1\le u\le v\le n$. That is, $S_{u,v}(x_1x_2\ldots x_n)$ swaps x_u and x_v of $x_1x_2\ldots x_n$. When $u=v$, $S_{u,v}(x_1x_2\ldots x_n)=x_1x_2\ldots x_n$. Since $d_H(G(i), G(j))=d_H(S_{u,v}(G(i)), S_{u,v}(G(j)))$ for any two code words $G(i)$ and $G(j)$, we have the following lemma.

Lemma 3. Suppose that $G(0)$, $G(1)$, …, $G(2^n-1)$ are a Gray code of n bits with the property (P2) (or (P3) or (P4)). Then, $S_{u,v}(G(0))$, $S_{u,v}(G(1))$, …, $S_{u,v}(G(2^n-1))$ are a Gray code of n bits with the property (P2) (or (P3) or (P4)) as well, where $1\le u\le v\le n$.

Similarly, we have the following lemma.

Lemma 4. Suppose that $G(0)$, $G(1)$, …, $G(2^n-1)$ are a Gray code of n bits with the property (P2) (or (P3) or (P4)), and Y is an n-bit binary sequence, where $n\ge 1$. Then, $Y\oplus G(0)$, $Y\oplus G(1)$, …, $Y\oplus G(2^n-1)$ are a Gray code of n bits with the property (P2) (or (P3) or (P4)) as well, where $\oplus$ performs a bitwise exclusive-OR operation.

3 The pancycle problem

In this section, the pancycle problem on the HCN(n) is explored. We show that the HCN(n) contains cycles of lengths ranging from 4 to 2^{2n}, exclusive of length 5. There is no cycle of length three or five in the HCN(n). Our results can be applied to the hierarchical folded-hypercube network (Duh and Chen 1995) which has a structure similar to the HCN(n).

Yun and Park (1995) showed that the HCN(n) contains cycles of even lengths ranging from 4 to 2^{2n}. Cycles of odd lengths can be constructed as follows. Suppose that $(X, Y_1)\rightarrow(X, Y_2)\rightarrow(X, Y_3)\rightarrow(X, Y_4)\rightarrow(X, Y_1)$ is an arbitrary cycle of length four in a cluster. There are the following cycles in the HCN(n).

$(Y_1, Y_3)\rightarrow(Y_3, Y_1)\Rightarrow(Y_3, Y_2)\rightarrow(Y_2, Y_3)\rightarrow(Y_2, Y_4)\rightarrow(Y_2, Y_1)\rightarrow(Y_1, Y_2)\Rightarrow(Y_1, Y_3)$.

By Lemma 1, $(Y_3, Y_1)\Rightarrow(Y_3, Y_2)$ and $(Y_1, Y_2)\Rightarrow(Y_1, Y_3)$ have lengths 1, 3, 5, …, 2^n-1. Hence these cycles have odd lengths ranging from 7 to $2^{n+1}+3$. On the other hand, using a reflected Gray code of $n-1$ bits, cycles of odd lengths ranging from $2^{n+1}+3$ to $2^{2n}-1$ can be obtained in the HCN(n) as follows.

$(0G^R(0), 0G^R(2^{n-1}-2))\Rightarrow(0G^R(0), 1G^R(1))$
$\rightarrow(1G^R(1), 0G^R(0))\Rightarrow(1G^R(1), 0G^R(1))$
$\rightarrow(0G^R(1), 1G^R(1))\Rightarrow(0G^R(1), 1G^R(2))$
$\rightarrow(1G^R(2), 0G^R(1))\Rightarrow(1G^R(2), 0G^R(2))$
$\rightarrow(0G^R(2), 1G^R(2))\Rightarrow(0G^R(2), 1G^R(3))$
$\rightarrow\ldots$
$\rightarrow(0G^R(2^{n-1}-3), 1G^R(2^{n-1}-3)\Rightarrow(0G^R(2^{n-1}-3), 1G^R(2^{n-1}-2)$
$\rightarrow(1G^R(2^{n-1}-2), 0G^R(2^{n-1}-3))\Rightarrow(1G^R(2^{n-1}-2), 0G^R(2^{n-1}-2))$
$\rightarrow(0G^R(2^{n-1}-2), 1G^R(2^{n-1}-2))\Rightarrow(0G^R(2^{n-1}-2), 1G^R(2^{n-1}-1))$
$\rightarrow(1G^R(2^{n-1}-1), 0G^R(2^{n-1}-2))\Rightarrow(1G^R(2^{n-1}-1), 0G^R(2^{n-1}-1))$
$\rightarrow(0G^R(2^{n-1}-1), (1G^R(2^{n-1}-1))\Rightarrow(0G^R(2^{n-1}-1), 1G^R(0))$
$\rightarrow(1G^R(0), 0G^R(2^{n-1}-1))\Rightarrow(1G^R(0), 0G^R(2^{n-1}-2))$
$\rightarrow(0G^R(2^{n-1}-2), 1G^R(0))\Rightarrow(0G^R(2^{n-1}-2), 0G^R(0))$
$\rightarrow(0G^R(0), 0G^R(2^{n-1}-2))$.

There are a total of 2^n+1 paths above, which are denoted by $\Rightarrow$. Figure 2 depicts the cycles in the HCN(3). The property (P3) can assure $d_H(G^R(1), G^R(2^{n-1}-2))=1$. Hence $d_H(0G^R(2^{n-1}-2), 1G^R(1))=2$. By Lemma 1, $(0G^R(0), 0G^R(2^{n-1}-2))\Rightarrow(0G^R(0), 1G^R(1))$ has lengths 2, 4, 6, …, 2^n-2. By Lemma 2, $(0G^R(2^{n-1}-2), 1G^R(2^{n-1}-2))\Rightarrow(0G^R(2^{n-1}-2), 1G^R(2^{n-1}-1))$ and $(0G^R(2^{n-1}-2), 1G^R(0))\Rightarrow(0G^R(2^{n-1}-2), 0G^R(0))$ can be made disjoint with lengths 1, 3, 5, …, $2^{n-1}-1$. The other paths have lengths 1, 3, 5, …, 2^n-1. Consequently, these cycles have odd lengths ranging from $2^{n+1}+3$ to $2^{2n}-1$.

There is no cycle of length three or five in the HCN(n), as explained below. Since the hypercube is bipartite, every cycle inside a cluster has even length. On the other hand, every cycle passing two clusters has length $2n+2$ at least because it contains one diameter link and one nondiame-

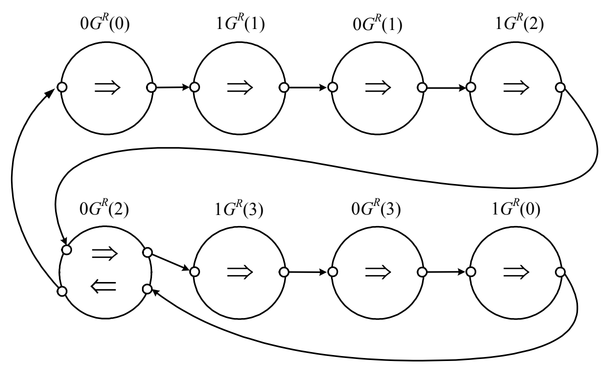

Figure 2. Cycles in the HCN(3) with lengths 19, 21, …,63.

ter link. Every cycle passing $r \geq 3$ clusters has length $2r \geq 6$ at least.

It is easy to see that the HCN(1) forms a cycle of length four and the HCN(2) contains cycles of lengths ranging from 4 to 16, exclusive of length 5. According to the discussion above, we have the following theorem, which solves the pancycle problem on the HCN(n).

Theorem 1. The HCN(n) contains cycles of lengths ranging from 4 to 2^{2n}, exclusive of length 5, where $n \geq 1$. There is no cycle of length three or five in the HCN(n).

Duh, Chen, and Fang (1995) proposed a two-level network, called the *hierarchical folded-hypercube network* (HFN), which is a modification of the HCN. We use HFN(n) to denote the HFN that contains 2^n clusters, where each cluster is an n-dimensional folded hypercube (n-fcube for short) (El-Amawy and S. Latifi 1991). The n-fcube is obtained by augmenting the n-cube with 2^{n-1} complement links. Each complement link connects two nodes whose addresses are the binary complement of each other. The HFN(n) can be obtained from the HCN(n) by removing the diameter links and replacing n-cubes with n-fcubes as clusters. The HFN(n) has the same node set as the HCN(n).

The HFN(1) forms a path of length three. It is not difficult to check that the HFN(2) contains cycles of lengths ranging from 3 to 2^{2n}, exclusive of length 5. When $n \geq 3$, the HFN(n) contains all the cycles that were obtained above for the HCN(n), because they do not contain diameter links. There is no cycle of length three contained in the HFN(n) if $n \geq 3$. The HFN(4) also contains a cycle of length five because such a cycle can be found in a 4-fcube. There is no cycle of length five contained in the HFN(n) if $n \neq 4$. We have the following corollary, which solves the pancycle problem on the HFN(n).

Corollary 1. The HFN(n) contains cycles of lengths ranging from 4 to 2^{2n}, exclusive of length 5, where $n \geq 2$. Moreover, the HFN(n) contains a cycle of length three (or five) if and only if $n=2$ (or $n=4$).

4 Hamiltonian-connectedness

In this section, a hamiltonian path is constructed between two arbitrary distinct nodes (X, Y) and (X', Y') of the HCN(n). The construction depends on whether (X, Y) and (X', Y') belong to the same cluster or not. Section 4.1 assumes that they belong to the same cluster. Section 4.2 assumes that they belong to two different clusters.

4.1 When (X, Y) and (X', Y') belong to the same cluster

We have $X=X'$. Suppose $X=x_1x_2\ldots x_n=X'$, $Y=y_1y_2\ldots y_n$, $Y'=y'_1y'_2\ldots y'_n$, $A=a_1a_2\ldots a_n$, and $A'=a'_1a'_2\ldots a'_n$. A hamiltonian (X, Y)-(X', Y') path can be obtained with the following three steps (refer to Figure 3).

Step 1. Determine $A \neq X$ and $A' \neq X'$ so that $d_H(Y, A)$ is odd, $d_H(Y', A')$ is odd, and $y_i = a_i = \overline{y'_i} = \overline{a'_i}$ for some $1 \leq i \leq n$.

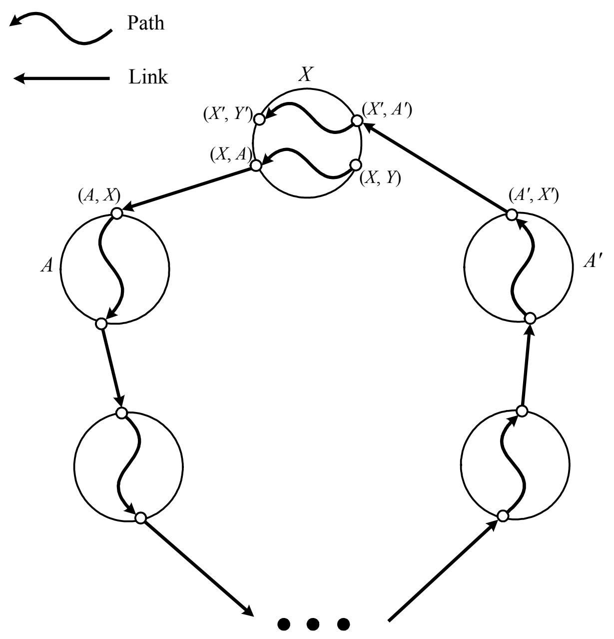

Figure 3. The construction of a hamiltonian (X, Y)-(X', Y') path in the HCN(n) when $X=X'$.

Step 2. Construct two disjoint (X, Y)-(X, A) and (X', A')-(X', Y') paths inside the cluster X that contain all nodes of the cluster X.

Step 3. Construct an (X, A)-(X', A') path in the HCN(n) that contains all nodes of the other clusters.

Step 1 is crucial to the success of Step 2 and Step 3. With A and A', Step 2 can be completed as follows. Since Y and A belong to $*^{i-1}y_i*^{n-i}$ and Y' and A' belong to $*^{i-1}\overline{y'_i}*^{n-i}$, by Lemma 1 there are an (X, Y)-(X, A) path of length $2^{n-1}-1$ in $*^{i-1}y_i*^{n-i}$ and an (X, Y')-(X, A') path of length $2^{n-1}-1$ in $*^{i-1}\overline{y'_i}*^{n-i}$. The two paths are disjoint, and they contain all nodes of the cluster X. On the other hand, Step 3 requires a Gray code $G(0)$, $G(1)$, …, $G(2^n-1)$ of n bits with some properties. By its aid, the (X, A)-(X', A') path can be obtained.

In the rest of this section, we focus our attention on how to determine A and A', how to generate $G(0)$, $G(1)$, …, $G(2^n-1)$, and how to construct the (X, A)-(X', A') path. Section 4.1.1 assumes $d_H(Y, Y')$ odd, and Section 4.1.2 assumes $d_H(Y, Y')$ even. Since $X=X'$, we have $Y \neq Y'$. Without loss of generality, we assume $y_1 \neq y'_1$ (i.e., $y_1 = \overline{y'_1}$). Besides, we assume $x_1=y_1$. If $x_1 \neq y_1$, then $x_1=y'_1$. The discussion for $x_1=y'_1$ is very similar to the discussion for $x_1=y_1$.

4.1.1 When $d_H(Y, Y')$ is odd

Suppose $Y^{(r)} \neq X$ for some $2 \leq r \leq n$. Let $A=Y^{(r)}$ and $A'=Y^{(r)(1)}$ ($\neq X'$), where $Y^{(r)(1)}$ is an adjacent node of $Y^{(r)}$ in an n-cube which differs from $Y^{(r)}$ at the first bit position. Then, $\overline{a'_1}=a_1=y_1=\overline{y'_1}$, $d_H(Y, A)=1$, $d_H(Y, A')=2$, and $d_H(Y', A')$ is odd (as a consequence of $d_H(Y, Y')$ odd and $d_H(Y, A')$ even).

$G(0)$, $G(1)$, …, $G(2^n-1)$ are required to have the following properties: $G(0)=A$, $G(2^{n-1}-1)=A'$, $X \in \{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$, and the properties (P2) and (P3). They can be generated as follows. Suppose $x_v \neq a_v$ (i.e., $x_v=\overline{a_v}$) for some $2 \leq v \leq n$ (recall $X \neq A$ and $x_1=y_1=a_1$). For all $0 \leq i \leq 2^n-1$, let $G(i)=A \oplus S_{1,2}(G^R(i))$ if $v=2$, and $G(i)=A \oplus S_{1,2}(S_{1,v}(G^R(i)))$ if $v>2$.

When $v=2$, $G(0)=A \oplus S_{1,2}(G^R(0))=A \oplus S_{1,2}(0^n)=A \oplus 0^n=A$ and $G(2^{n-1}-1)=A \oplus S_{1,2}(G^R(2^{n-1}-1))=A \oplus S_{1,2}(010^{n-2})=A \oplus 10^{n-1}=A^{(1)}$. Since $A=Y^{(r)}$ and $A'=Y^{(r)(1)}$, we have $G(2^{n-1}-1)=A^{(1)}=A'$. By the property (P1), the second bit of $S_{1,2}(G^R(j))$ is 0 if $0 \leq j \leq 2^{n-1}-1$, and 1 if $2^{n-1} \leq j \leq 2^n-1$. Hence the second bit of $G(j)=A \oplus S_{1,2}(G^R(j))$ is a_2 if $0 \leq j \leq 2^{n-1}-1$, and $\overline{a_2}$ if $2^{n-1} \leq j \leq 2^n-1$. Since $x_2=\overline{a_2}$, we have $X \in \{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$. The Gray code has the properties (P2) and (P3), as a consequence of Lemma 3 and Lemma 4. When $v>2$, we have $G(0)=A$, $G(2^{n-1}-1)=A'$, and $X \in \{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$, similarly. Besides, the properties (P2) and (P3) hold.

Suppose $X=G(2^{n-1}+m)$, where $0 \leq m \leq 2^{n-1}-1$. Define $G'(j)=G(2^{n-1}+((m+j) \bmod 2^{n-1}))$ for all $0 \leq j \leq 2^{n-1}-1$. The mapping from $\{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$ to $\{G'(0), G'(1), \ldots, G'(2^{n-1}-1)\}$ is shown in Figure 4.

The following is an (X, A)-(X', A') path in the HCN(n).

(X, A) (=$(G'(0), G(0))$)
$\rightarrow (G(0), G'(0)) \Rightarrow_H (G(0), G'(1))$
$\rightarrow (G'(1), G(0)) \Rightarrow_H (G'(1), G(1))$
$\rightarrow (G(1), G'(1)) \Rightarrow_H (G(1), G'(2))$
$\rightarrow (G'(2), G(1)) \Rightarrow_H (G'(2), G(2))$
$\rightarrow \ldots$
$\rightarrow (G(2^{n-1}-1), G'(2^{n-1}-1)) \Rightarrow_H (G(2^{n-1}-1), G'(0))$
$\rightarrow (G'(0), G(2^{n-1}-1))$ (=(X, A')),

where $\Rightarrow_H$ denotes a hamiltonian path for a cluster.

The (X, A)-(X', A') path traverses clusters $G(0)$, $G'(1)$, $G(1)$, …, $G'(2^{n-1}-1)$, $G(2^{n-1}-1)$, sequentially. By the properties (P2) and (P3), we have $d_H(G(i), G((i+1) \bmod 2^{n-1}))=1$ and $d_H(G'(i), G'((i+1) \bmod 2^{n-1}))=1$ for all $0 \leq i \leq 2^{n-1}-1$. Lemma 1 can assure the existence of the hamiltonian paths denoted by $\Rightarrow_H$. The (X, A)-(X', A') path contains all nodes of the clusters, exclusive of $G'(0)$ (=X).

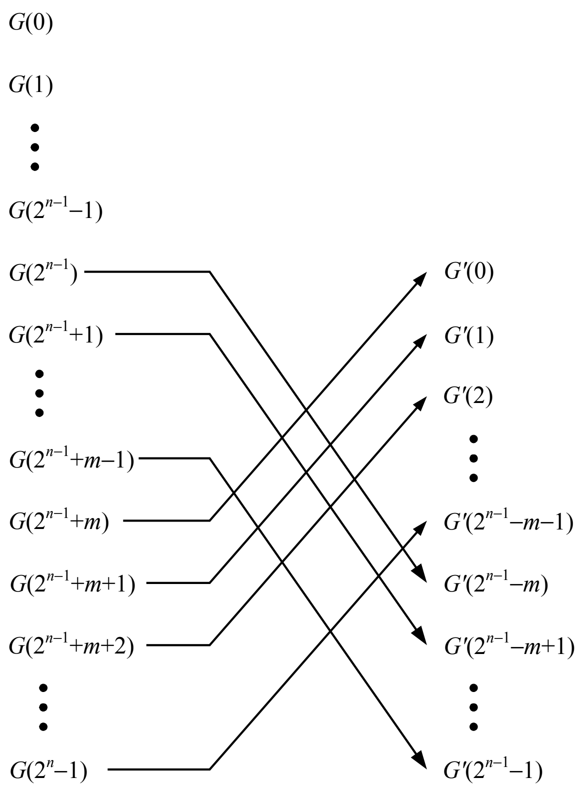

Figure 4. The mapping from $\{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$ to $\{G'(0), G'(1), \ldots, G'(2^{n-1}-1)\}$ when $d_H(Y, Y')$ is odd.

4.1.2 When $d_H(Y, Y')$ is even

Both $d_H(X, Y)$ and $d_H(X, Y')$ are odd or even, as a consequence of $d_H(Y, Y')$ even. Two cases are discussed below.

Case 1. $d_H(X, Y)$ and $d_H(X, Y')$ are odd. Let $A=X^{(n)(2)}$ and $A'=X^{(n)(1)}$. Then, $\overline{a'_1}=a_1=x_1=y_1=\overline{y'_1}$. We have $d_H(Y, A)$ odd because $d_H(X, Y)$ is odd and $d_H(X, A)=2$. We have $d_H(Y', A')$ odd, similarly.

$G(0)$, $G(1)$, …, $G(2^n-1)$ are required to have the following properties: $G(0)=X$, $G(2^{n-1}-2)=A'$, $G(2^n-2)=A$, and the properties (P2) and (P3). They can be generated as follows: $G(i)=X \oplus S_{1,2}(G^R(i))$ for all $0 \leq i \leq 2^n-1$. It is not difficult to verify $G(0)=X$, $G(2^{n-1}-2)=A'$, and $G(2^n-2)=A$. They have the properties (P2) and (P3), as a consequence of Lemma 3 and Lemma 4.

Define $G'(j)=G(2^n-1-j)$ for all $0 \leq j \leq 2^{n-1}-1$. The mapping from $\{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$ to $\{G'(0), G'(1),$

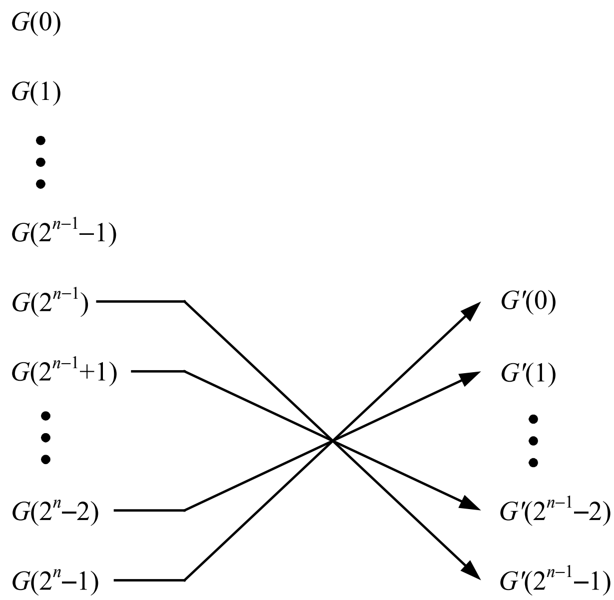

Figure 5. The mapping from $\{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$ to $\{G'(0), G'(1), \ldots, G'(2^{n-1}-1)\}$ when $d_H(X, Y)$ and $d_H(X, Y')$ are odd.

$\ldots, G'(2^{n-1}-1)\}$ is shown in Figure 5.

The following is an (X, A)-(X', A') path in the HCN(n).

(X, A) $(=(G(0), G(2^n-2))=(G(0), G'(1)))$
$\rightarrow (G'(1), G(0)) \Rightarrow_H (G'(1), G(1))$
$\rightarrow (G(1), G'(1)) \Rightarrow_H (G(1), G'(2))$
$\rightarrow (G'(2), G(1)) \Rightarrow_H (G'(2), G(2))$
$\rightarrow \ldots$
$\rightarrow (G(2^{n-1}-3), G'(2^{n-1}-3)) \Rightarrow_H (G(2^{n-1}-3), G'(2^{n-1}-2))$
$\rightarrow (G'(2^{n-1}-2), G(2^{n-1}-3)) \Rightarrow_H (G'(2^{n-1}-2), G(2^{n-1}-2))$
$\rightarrow (G(2^{n-1}-2), G'(2^{n-1}-2)) \Rightarrow (G(2^{n-1}-2), G'(2^{n-1}-1))$
$\rightarrow (G'(2^{n-1}-1), G(2^{n-1}-2)) \Rightarrow_H (G'(2^{n-1}-1), G(2^{n-1}-1)$
$\rightarrow (G(2^{n-1}-1), G'(2^{n-1}-1)) \Rightarrow_H (G(2^{n-1}-1), G'(0))$
$\rightarrow (G'(0), G(2^{n-1}-1)) \Rightarrow_H (G'(0), G(2^{n-1}-2))$
$\rightarrow (G(2^{n-1}-2), G'(0)) \Rightarrow (G(2^{n-1}-2), G(0))$
$\rightarrow (G(0), G(2^{n-1}-2))$ $(=(X, A'))$,

where $(G(2^{n-1}-2), G'(2^{n-1}-2)) \Rightarrow (G(2^{n-1}-2), G'(2^{n-1}-1))$ and $(G(2^{n-1}-2), G'(0)) \Rightarrow (G(2^{n-1}-2), G(0))$ are two disjoint paths of length $2^{n-1}-1$ in the cluster $G(2^{n-1}-2)$.

The (X, A)-(X', A') path traverses clusters $G'(1)$, $G(1)$, $G'(2)$, $G(2)$, $\ldots$, $G'(2^{n-1}-2)$, $G(2^{n-1}-2)$, $G'(2^{n-1}-1)$, $G(2^{n-1}-1)$, $G'(0)$, $G(2^{n-1}-2)$, sequentially ($G(2^{n-1}-2)$ is traversed twice). By the properties (P2) and (P3), we have $d_H(G(i), G'(i))=1$, $d_H(G(i), G((i+1) \bmod 2^{n-1}))=1$, and $d_H(G'(i), G'((i+1) \bmod 2^{n-1}))=1$ for all $0 \le i \le 2^{n-1}-1$. Lemma 1 can assure the existence of the hamiltonian paths denoted by $\Rightarrow_H$. Lemma 2 can assure the existence of the two disjoint paths in the cluster $G(2^{n-1}-2)$. The (X, A)-(X', A') path contains all nodes of the clusters, exclusive of $G(0)$ $(=X)$. Figure 6 shows a hamiltonian (100, 011)-(100, 101) path in the HCN(3).

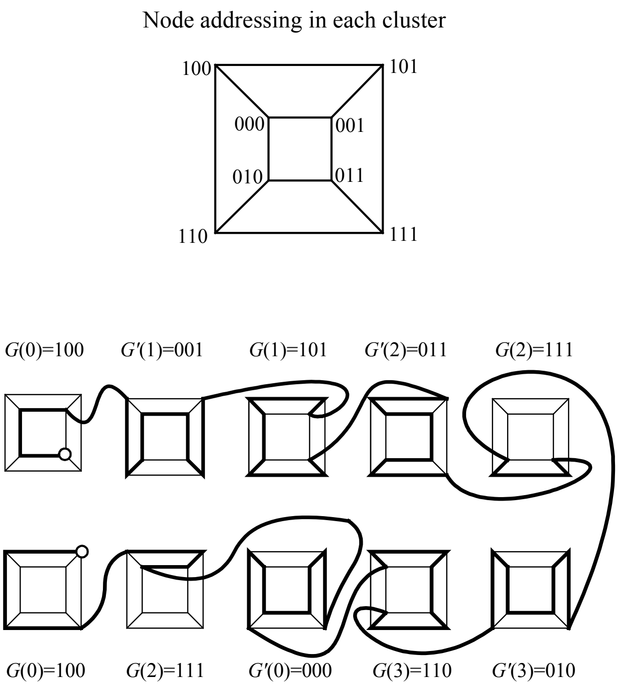

Figure 6. A hamiltonian (100, 011)-(100, 101) path in the HCN(3).

Case 2. $d_H(X, Y)$ and $d_H(X, Y')$ are even. Let $A=X^{(2)}$ and $A'=X^{(1)}$. Then, $\overline{a'_1}=a_1=x_1=y_1=\overline{y'_1}$. We have $d_H(Y, A)$ odd because $d_H(X, Y)$ is even and $d_H(X, A)=1$. We have $d_H(Y', A')$ odd, similarly.

$G(0), G(1), \ldots, G(2^n-1)$ are required to have the following properties: $G(0)=X$, $G(2^{n-1}-1)=A$, $G(2^n-1)=A'$, and the properties (P2) and (P3). They can be generated as follows: $G(i)=X \oplus G^R(i)$ for all $0 \le i \le 2^n-1$. It is not difficult to verify $G(0)=X$, $G(2^{n-1}-1)=A$, and $G(2^n-1)=A'$. They have the properties (P2) and (P3), because of Lemma 4.

Define $G'(0), G'(1), \ldots, G'(2^{n-1}-1)$ all the same as Case 1. The following is an (X, A)-(X', A') path in the HCN(n).

(X, A) $(=(G(0), G(2^{n-1}-1)))$
$\rightarrow (G(2^{n-1}-1), G(0)) \Rightarrow (G(2^{n-1}-1), G(1))$
$\rightarrow (G(1), G(2^{n-1}-1)) \Rightarrow (G(1), G(2^{n-1}-2))$
$\rightarrow (G(2^{n-1}-2), G(1)) \Rightarrow (G(2^{n-1}-2), G'(1))$

$\rightarrow (G'(1), G(2^{n-1}-2)) \Rightarrow_H (G'(1), G(1))$
$\rightarrow (G(1), G'(1)) \Rightarrow (G(1), G'(2))$
$\rightarrow (G'(2), G(1)) \Rightarrow_H (G'(2), G(2))$
$\rightarrow (G(2), G'(2)) \Rightarrow_H (G(2), G'(3))$
$\rightarrow \ldots$
$\rightarrow (G(2^{n-1}-3), G'(2^{n-1}-3)) \Rightarrow_H (G(2^{n-1}-3), G'(2^{n-1}-2))$
$\rightarrow (G'(2^{n-1}-2), G(2^{n-1}-3)) \Rightarrow_H (G'(2^{n-1}-2), G(2^{n-1}-2))$
$\rightarrow (G(2^{n-1}-2), G'(2^{n-1}-2)) \Rightarrow (G(2^{n-1}-2), G'(2^{n-1}-1))$
$\rightarrow (G'(2^{n-1}-1), G(2^{n-1}-2)) \Rightarrow_H (G'(2^{n-1}-1), G(2^{n-1}-1))$
$\rightarrow (G(2^{n-1}-1), G'(2^{n-1}-1)) \Rightarrow (G(2^{n-1}-1), G'(0))$
$\rightarrow (G'(0), G(2^{n-1}-1)) \Rightarrow_H (G'(0), G(0))$
$\rightarrow (G(0), G'(0))$ $(=(G(0), G(2^n-1))=(X, A'))$

The (X, A)-(X', A') path traverses clusters $G(2^{n-1}-1)$, $G(1)$, $G(2^{n-1}-2)$, $G'(1)$, $G(1)$, $G'(2)$, $G(2)$, $\ldots$, $G(2^{n-1}-3)$, $G'(2^{n-1}-2)$, $G(2^{n-1}-2)$, $G'(2^{n-1}-1)$, $G(2^{n-1}-1)$, $G'(0)$, sequentially ($G(1)$, $G(2^{n-1}-2)$, and $G(2^{n-1}-1)$ are traversed twice). There are $(G'(u), G(u-1)) \Rightarrow_H (G'(u), G(u))$ for all $2 \le u \le 2^{n-1}-2$, $(G(v), G'(v)) \Rightarrow_H (G(v), G'(v+1))$ for all $2 \le v \le 2^{n-1}-3$, and two disjoint paths of length $2^{n-1}-1$ in the clusters $G(1)$, $G(2^{n-1}-2)$, and $G(2^{n-1}-1)$. Similarly, by the aid of the properties (P2) and (P3), Lemma 1, and Lemma 2, the (X, A)-(X', A') path contains all nodes of the clusters, exclusive of $G(0)$ $(=X)$. Figure 7 shows a hamiltonian (100, 111)-(100, 010) path in the HCN(3).

4.2 When (X, Y) and (X', Y') belong to different clusters

We have $X \neq X'$. A hamiltonian (X, Y)-(X', Y') path can be obtained with the following three steps (refer to Figure 8).

Step 1. Determine $A \neq X$ and $A' \neq X'$ so that $d_H(Y, A)$ is odd and $d_H(Y', A')$ is odd.

Step 2. Construct a hamiltonian (X, Y)-(X, A) path for the cluster X and a hamiltonian (X', A')-(X', Y') path for the cluster X'.

Step 3. Construct an (X, A)-(X', A') path in the HCN(n) that contains all nodes of the other clusters.

Different A and A' are needed for different cases. Since $d_H(Y, A)$ and $d_H(Y', A')$ are odd, by Lemma 1 there are a hamiltonian (X, Y)-(X, A) path for the cluster X and a

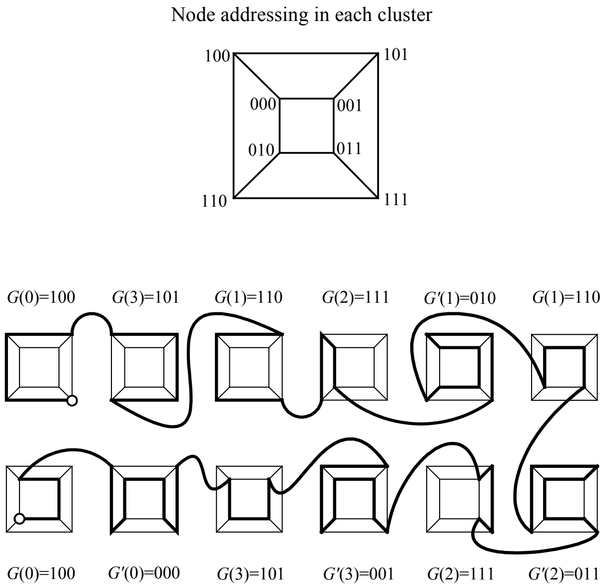

Figure 7. A hamiltonian (100, 111)-(100, 010) path in the HCN(3).

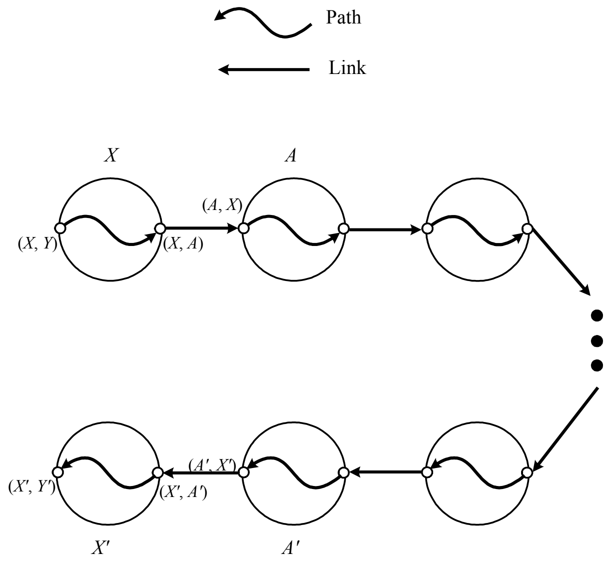

Figure 8. The construction of a hamiltonian (X, Y)-(X', Y') path in the HCN(n) when $X \neq X'$.

hamiltonian (X', A')-(X', Y') path for the cluster X'. The construction of the (X, A)-(X', A') path requires a Gray code $G(0), G(1), \ldots, G(2^n-1)$ of n bits with the following properties: $G(0)=X$, $X' \in \{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$, and the properties (P2), (P3), and (P4). They can be generated as follows. Suppose $x_u \neq x'_u$, where $X=x_1x_2\ldots x_n$, $X'=x'_1x'_2\ldots x'_n$, and $1 \leq u \leq n$. Let $G(i)=X \oplus S_{1,u}(G^R(i))$ for all $0 \leq i \leq 2^n-1$. It is not difficult to verify $G(0)=X$ and $X' \in \{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$. Besides, they have the properties (P2), (P3), and (P4), because of Lemma 3 and Lemma 4.

In the rest of this section, we focus our attention on how to determine A and A' and how to construct the (X, A)-(X', A') path. Subsequent discussion depends on whether $d_H(X, Y')$ is even or odd and whether $d_H(X', Y)$ is even or odd.

Case 1. $d_H(X, Y')$ is even and $d_H(X', Y)$ is even. Suppose $X'=G(2^{n-1}+m)$, where $0 \leq m \leq 2^{n-1}-1$. Define $G'(j)$ all the same as Section 4.1.1 (refer to Figure 4). Then $X'=G'(0)$. Let $A=G'(1)$ and $A'=G(2^{n-1}-1)$. We have $d_H(Y, A)$ odd because $d_H(X', Y)$ is even and $d_H(X', A)=d_H(G'(0), G'(1))=1$. We have $d_H(Y', A')$ odd, similarly. The following is an (X, A)-(X', A') path in the HCN(n).

(X, A) $(=(G(0), G'(1)))$
$\rightarrow (G'(1), G(0)) \Rightarrow_H (G'(1), G(1))$
$\rightarrow (G(1), G'(1)) \Rightarrow_H (G(1), G'(2))$
$\rightarrow \ldots$
$\rightarrow (G(2^{n-1}-1), G'(2^{n-1}-1)) \Rightarrow_H (G(2^{n-1}-1), G'(0))$
$\rightarrow (G'(0), G(2^{n-1}-1))$ $(=(X', A'))$

The (X, A)-(X', A') path traverses clusters $G'(1), G(1), \ldots, G'(2^{n-1}-1), G(2^{n-1}-1)$, sequentially. Similarly, by the aid of the properties (P2) and (P3) and Lemma 1, the (X, A)-(X', A') path contains all nodes of the clusters, exclusive of $G(0)$ $(=X)$ and $G'(0)$ $(=X')$.

Case 2. $d_H(X, Y')$ is odd and $d_H(X', Y)$ is odd. Suppose $X'=G(2^{n-1}+m)$, where $0 \leq m \leq 2^{n-1}-1$. Define $G'(0), G'(1), \ldots, G'(2^{n-1}-1)$ all the same as Case 1. We have $X'=G'(0)$. Let $A=G'(2)$ and $A'=G(2^{n-1}-2)$. We have $d_H(X', A)=d_H(G'(0), G'(2))=2$. By the property (P4), $d_H(X, A')=d_H(G(0), G(2^{n-1}-2))=2$. Hence both $d_H(Y, A)$ and $d_H(Y', A')$ are odd because $d_H(X', Y)$ is odd and $d_H(X, Y')$ is odd. The following is an (X, A)-(X', A') path in the HCN(n).

(X, A) $(=(G(0), G'(2)))$
$\rightarrow (G'(2), G(0)) \Rightarrow (G'(2), G(1))$
$\rightarrow (G(1), G'(2)) \Rightarrow_H (G(1), G'(1))$
$\rightarrow (G'(1), G(1)) \Rightarrow_H (G'(1), G(2))$
$\rightarrow (G(2), G'(1)) \Rightarrow_H (G(2), G'(2))$
$\rightarrow (G'(2), G(2)) \Rightarrow (G'(2), G(3))$
$\rightarrow (G(3), G'(2)) \Rightarrow_H (G(3), G'(3))$
$\rightarrow (G'(3), G(3)) \Rightarrow_H (G'(3), G(4))$
$\rightarrow \ldots$
$\rightarrow (G(2^{n-1}-3), G'(2^{n-1}-4)) \Rightarrow_H (G(2^{n-1}-3), G'(2^{n-1}-3))$
$\rightarrow (G'(2^{n-1}-3), G(2^{n-1}-3)) \Rightarrow_H (G'(2^{n-1}-3), G(2^{n-1}-2))$
$\rightarrow (G(2^{n-1}-2), G'(2^{n-1}-3)) \Rightarrow (G(2^{n-1}-2), G'(2^{n-1}-2))$
$\rightarrow (G'(2^{n-1}-2), G(2^{n-1}-2)) \Rightarrow_H (G'(2^{n-1}-2), G(2^{n-1}-1))$
$\rightarrow (G(2^{n-1}-1), G'(2^{n-1}-2)) \Rightarrow_H (G(2^{n-1}-1), G'(2^{n-1}-1))$
$\rightarrow (G'(2^{n-1}-1), G(2^{n-1}-1)) \Rightarrow_H (G'(2^{n-1}-1), G(2^{n-1}-2))$
$\rightarrow (G(2^{n-1}-2), G'(2^{n-1}-1)) \Rightarrow (G(2^{n-1}-2), G'(0))$
$\rightarrow (G'(0), G(2^{n-1}-2))$ $(=(X', A'))$

The (X, A)-(X', A') path traverses clusters $G'(2), G(1), G'(1), G(2), G'(2), G(3), G'(3), \ldots, G(2^{n-1}-3), G'(2^{n-1}-3), G(2^{n-1}-2), G'(2^{n-1}-2), G(2^{n-1}-1), G'(2^{n-1}-1), G(2^{n-1}-2)$, sequentially ($G'(2)$ and $G(2^{n-1}-2)$ are traversed twice). There are $(G(u), G'(u-1)) \Rightarrow_H (G(u), G'(u))$ and $(G'(u), G(u)) \Rightarrow_H (G'(u), G'(u+1))$ for all $3 \leq u \leq 2^{n-1}-3$, and two disjoint paths of length $2^{n-1}-1$ in the clusters $G'(2)$ and $G(2^{n-1}-2)$. Similarly, the (X, A)-(X', A') path contains all nodes of the clusters, exclusive of $G(0)$ $(=X)$ and $G'(0)$ $(=X')$. Figure 9 shows a hamiltonian (100, 100)-(110, 101) path in the HCN(3).

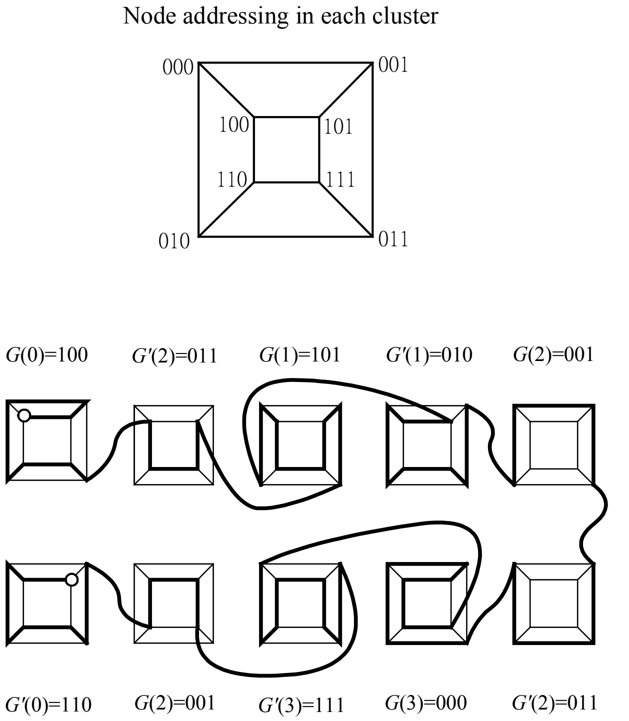

Figure 9. A hamiltonian (100, 100)-(110, 101) path in the HCN(3).

Case 3. $d_H(X, Y')$ is even and $d_H(X', Y)$ is odd. Two cases, $d_H(X, X')$ odd or even, need to be discussed below.

Case 3.1. $d_H(X, X')$ is odd. Let $A=A'=G(2^{n-1}-1)$. We have $d_H(X', A)$ even because $d_H(X, X')$ is odd and $d_H(X, A)= d_H(G(0), G(2^{n-1}-1))=1$ (by the property (P3)). Hence $d_H(Y, A)$ is odd, as a consequence of $d_H(X', Y)$ odd and $d_H(X', A)$ even. Similarly, $d_H(Y', A')$ is odd.

Recall $X' \in \{G(2^{n-1}), G(2^{n-1}+1), \ldots, G(2^n-1)\}$ and $G(i)= X\oplus S_{1,u}(G^R(i))$, where $x_u \neq x'_u$. Clearly, the uth bit of $G(2^{n-1})$, $G(2^{n-1}+1)$, …, $G(2^n-1)$ is $\overline{x_u}$. In other words, $\{G(2^{n-1})$, $G(2^{n-1}+1)$, …, $G(2^n-1)\}$ constitutes the node set of $*^{u-1}\,\overline{x_u}\,*^{n-u}$. Since $d_H(X, G(2^n-2))=d_H(G(0), G(2^n-2))=2$ (by the property (P4)) and $d_H(X, X')$ is odd, we have $d_H(G(2^n-2), X')$ odd. By Lemma 1, there is a hamiltonian $G(2^n-2)$-X' path for $*^{u-1}\,\overline{x_u}\,*^{n-u}$.

Define $G'(i)$ to be the $(i+1)$th node in the hamiltonian $G(2^n-2)$-X' path, where $0\le i\le 2^{n-1}-1$. That is, $G'(0)$ $(=G(2^n-2)) \to G'(1) \to G'(2) \to \ldots \to G'(2^{n-1}-1)$ $(=X')$ is the hamiltonian $G(2^n-2)$-X' path. The following is an (X, A)-(X', A') path in the HCN(n).

(X, A) $(=(G(0), G(2^{n-1}-1)))$
$\to (G(2^{n-1}-1), G(0)) \Rightarrow (G(2^{n-1}-1), G(1))$
$\to (G(1), G(2^{n-1}-1)) \Rightarrow (G(1), G(2^{n-1}-2))$
$\to (G(2^{n-1}-2), G(1)) \Rightarrow (G(2^{n-1}-2), G'(0))$
$\to (G'(0), G(2^{n-1}-2)) \Rightarrow_H (G'(0), G(1))$
$\to (G(1), G'(0)) \Rightarrow (G(1), G'(1))$
$\to (G'(1), G(1)) \Rightarrow_H (G'(1), G(2))$
$\to (G(2), G'(1)) \Rightarrow_H (G(2), G'(2))$
$\to \ldots$
$\to (G(2^{n-1}-3), G'(2^{n-1}-4)) \Rightarrow_H (G(2^{n-1}-3), G'(2^{n-1}-3))$
$\to (G'(2^{n-1}-3), G(2^{n-1}-3)) \Rightarrow_H (G'(2^{n-1}-3), G(2^{n-1}-2))$
$\to (G(2^{n-1}-2), G'(2^{n-1}-3)) \Rightarrow (G(2^{n-1}-2), G'(2^{n-1}-2))$
$\to (G'(2^{n-1}-2), G(2^{n-1}-2)) \Rightarrow_H (G'(2^{n-1}-2), G(2^{n-1}-1))$
$\to (G(2^{n-1}-1), G'(2^{n-1}-2)) \Rightarrow (G(2^{n-1}-1), G'(2^{n-1}-1))$
$\to (G'(2^{n-1}-1), G(2^{n-1}-1))$ $(=(X', A'))$

The (X, A)-(X', A') path traverses clusters $G(2^{n-1}-1)$, $G(1)$, $G(2^{n-1}-2)$, $G'(0)$, $G(1)$, $G'(1)$, $G(2)$, …, $G(2^{n-1}-3)$, $G'(2^{n-1}-3)$, $G(2^{n-1}-2)$, $G'(2^{n-1}-2)$, $G(2^{n-1}-1)$, sequentially ($G(1)$, $G(2^{n-1}-2)$, and $G(2^{n-1}-1)$ are traversed twice). There are $(G'(u), G(u)) \Rightarrow_H (G'(u), G(u+1))$ for all $1\le u\le 2^{n-1}-3$, $(G(v), G'(v-1)) \Rightarrow_H (G(v), G'(v))$ for all $2\le v\le 2^{n-1}-3$, and two disjoint paths of length $2^{n-1}-1$ in the clusters $G(1)$, $G(2^{n-1}-2)$, and $G(2^{n-1}-1)$. Similarly, the (X, A)-(X', A') path contains all nodes of the clusters, exclusive of $G(0)$ $(=X)$ and $G'(2^{n-1}-1)$ $(=X')$. Figure 10 shows a hamiltonian (110, 110)-(001, 101) path in the HCN(3).

Case 3.2. $d_H(X, X')$ is even. Let $A=G(2^{n-1}-2)$ and $A'=G(2^{n-1}-1)$. We have $d_H(X, A)=d_H(G(0), G(2^{n-1}-2))= 2$ by the property (P4). $d_H(Y, A)$ is odd, as a consequence of $d_H(X, A)$ even, $d_H(X, X')$ even, and $d_H(X', Y)$ odd. Similarly, $d_H(Y', A')$ is odd.

Since $d_H(X, G(2^n-1))=d_H(G(0), G(2^n-1))=1$ and $d_H(X, X')$ is even, we have $d_H(G(2^n-1), X')$ odd. Similar to Case 3.1, there is a hamiltonian $G(2^n-1)$-X' path for $*^{u-1}\,\overline{x_u}\,*^{n-u}$. Define $G'(0)$, $G'(1)$, …, $G'(2^{n-1}-1)$ all the same as Case 3.1. That is, $G'(0)$ $(=G(2^n-1)) \to G'(1) \to G'(2) \to \ldots \to G'(2^{n-1}-1)$ $(=X')$ is the hamiltonian $G(2^n-1)$-X' path. The following is an (X, A)-(X', A') path in the HCN(n).

(X, A) $(=(G(0), G(2^{n-1}-2)))$
$\to (G(2^{n-1}-2), G(0)) \Rightarrow (G(2^{n-1}-2), G'(0))$
$\to (G'(0), G(2^{n-1}-2)) \Rightarrow_H (G'(0), G(1))$
$\to (G(1), G'(0)) \Rightarrow_H (G(1), G'(1))$
$\to (G'(1), G(1)) \Rightarrow_H (G'(1), G(2))$
$\to \ldots$
$\to (G(2^{n-1}-3), G'(2^{n-1}-4)) \Rightarrow_H (G(2^{n-1}-3), G'(2^{n-1}-3))$
$\to (G'(2^{n-1}-3), G(2^{n-1}-3)) \Rightarrow_H (G'(2^{n-1}-3), G(2^{n-1}-2))$
$\to (G(2^{n-1}-2), G'(2^{n-1}-3)) \Rightarrow (G(2^{n-1}-2), G'(2^{n-1}-2))$
$\to (G'(2^{n-1}-2), G(2^{n-1}-2)) \Rightarrow_H (G'(2^{n-1}-2), G(2^{n-1}-1))$
$\to (G(2^{n-1}-1), G'(2^{n-1}-2)) \Rightarrow_H (G(2^{n-1}-1), G'(2^{n-1}-1))$
$\to (G'(2^{n-1}-1), G(2^{n-1}-1))$ $(=(X', A'))$.

where $(G(2^{n-1}-2), G(0)) \Rightarrow (G(2^{n-1}-2), G'(0))$ and $(G(2^{n-1}-2), G'(2^{n-1}-3)) \Rightarrow (G(2^{n-1}-2), G'(2^{n-1}-2))$ are two disjoint paths of length $2^{n-1}-1$ in the cluster $G(2^{n-1}-2)$. The (X, A)-(X', A') path traverses clusters $G(2^{n-1}-2)$, $G'(0)$, $G(1)$, $G'(1)$, …, $G(2^{n-1}-3)$, $G'(2^{n-1}-3)$, $G(2^{n-1}-2)$, $G'(2^{n-1}-2)$, $G(2^{n-1}-1)$, sequentially ($G(2^{n-1}-2)$ is traversed twice). Similarly, the (X, A)-(X', A') path contains all nodes of the clusters, exclusive of $G(0)$ $(=X)$ and $G'(2^{n-1}-1)$ $(=X')$.

Case 4. $d_H(X, Y')$ is odd and $d_H(X', Y)$ is even. Similar to Case 3.

The following theorem summarizes the main result of this section.

Theorem 2. The HCN(n) is hamiltonian-connected, where $n\ge 3$.

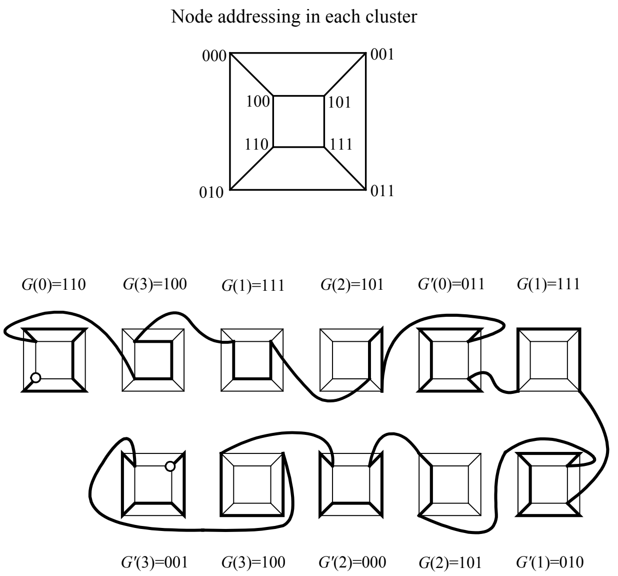

Figure 10. A hamiltonian (110, 110)-(001, 101) path in the HCN(3).

We note that no diameter link is contained in the hamiltonian (X, Y)-(X', Y') paths that we obtained in this section. It is easy to check that the HFN(2) is hamiltonian-connected. Therefore, we have the following corollary.

Corollary 2. The HFN(n) is hamiltonian-connected, where $n \geq 2$.

5 Discussion and conclusion

The HCN was proposed as an alternative to the hypercube. Although the HCN uses about half links of a comparable hypercube, its diameter, wide-diameter, and fault diameter are all about two-thirds those of a comparable hypercube. In this paper, two problems related to the hamiltonicity of the HCN were solved. First we solved the pancycle problem by showing that there are cycles of length l in the HCN(n) if and only if $4 \leq l \leq 2^{2n}$ and $l \neq 5$, where $n \geq 1$. In contrast, there are cycles of length l in a $2n$-cube if and only if $4 \leq l \leq 2^{2n}$ and l is even.

Second, we showed that the HCN(n) is hamiltonian-connected, where $n \geq 3$. That is, there is a hamiltonian path (of length $2^{2n}-1$) between every two distinct nodes of the HCN(n). On the other hand, Malluhi and Bayoumi showed that the hypercube is hamiltonian-laceable. Lemma 1 improved their work by showing that the hypercube is strongly hamiltonian-laceable. That is, between any two distinct nodes X and Y of a $2n$-cube, there is a hamiltonian X-Y path (of length $2^{2n}-1$) if $d_H(X, Y)$ is odd, and a (longest) X-Y path of length $2^{2n}-2$ if $d_H(X, Y)$ is even. With our results, it can be concluded that the HCN is superior to a comparable hypercube in hamiltonicity.

As a by-product, the two problems were solved on the HFN as well. It was shown that there are cycles of length l in the HFN(n) if and only if $4 \leq l \leq 2^{2n}$ and $l \neq 5$, where $n \geq 2$. Besides, the HFN(n) is hamiltonian-connected, where $n \geq 2$. Our results reveal that both the HCN and the HFN can embed a longest linear array between every two distinct nodes and rings of all possible lengths except three and five, with dilation, congestion, load, and expansion equal to one. Linear arrays and rings are two of the most fundamental networks for parallel and distributed computing. There are many efficient algorithms designed on them for solving a variety of problems (refer to Akl 1997 and Leighton 1992). These parallel algorithms can be executed on the HCN and the HFN as well.

There is a sufficient condition for a hamiltonian-connected graph as follows (Buckley and Harary 1990). If $G=(V, E)$ is a connected graph with $\deg(u)+\deg(v) \geq |V|+1$ for any two nonadjacent nodes u and v, then G is hamiltonian-connected, where $\deg(u)$ and $\deg(v)$ denote the degrees of u and v, respectively. The HCN is hamiltonian-connected, even if $\deg(u)+\deg(v)=2n+2<2^{2n}+1=|V|+1$ for the HCN(n). One of our further research topics is to explore the hamiltonicity of the HCN when there are node faults and/or edge faults.

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